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Finite Element Prediction of Damping in Structures with Constrained Viscoelastic Layers

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An efficient method is described for finite element modeling of three-layer laminates containing a viscoelastic layer. Modal damping ratios are estimated from undamped normal mode results by means of the modal strain energy (MSE) method. Comparisons are given between results obtained by the MSE method implemented in NASTRAN, by various exact solutions for approximate governing differential equations, and by experiment. Results are in terms of frequencies, modal damping ratios, and mechanical admittances for simple beams, plates, and rings. Application of the finite element-MSE method in design of integrally damped structures is discussed.

Introduction

THE use of distributed viscoelastic material in aerospace structures has a long and successful history as a means of controlling resonant vibration. One of the most weight-effective methods of incorporating a viscoelastic material in a built-up structure is in the form of a constrained layer. The elastomer is sandwiched between two metallic sheets and is bonded to both. Flexural vibration causes shearing strain in the core which dissipates energy and thereby reduces vibration.

Numerous analyses of simple sandwich structures (beams, plates, cylinders, etc.) have been published.[1-5] The usual approach is to begin with partial differential equations of motion that are derived by consideration of a thicknesswise element of the sandwich. Shearing is assumed to be the only significant energy storage mechanism in the core, and dissipation under harmonic loading is introduced by taking the core shear modulus to be complex. Even for simple geometries, algebraic and/or numerical solution of the equations of motion tends to be lengthy. In real-life situations, a designer is faced with constraints that limit the usefulness of purely analytical solutions. Structural geometries are usually complicated, and use of a layered damper is feasible in only certain areas of built-up assemblages. Design time schedules are tight, so a practical analysis method must not only give damping estimates of useful accuracy, but also give some indication of what should be changed in order to improve a candidate design.

Since much of the difficulty of designing layered dampers stems from complicated geometries, it is natural to look to finite element methods for solutions, just as they are used for analysis of general undamped structures. In this paper, several approaches to damped structural design are reviewed in the context of implementation by existing general-purpose finite element codes. One technique in particular, the modal strain energy method, is discussed with examples since it is believed to hold the greatest potential for design work. A modeling method is described that is suitable for three-layer sandwiches and can be readily accommodated within an existing commercial finite element program.

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Discretized Equations of Motion

Three distinct questions must be answered in arriving at response predictions by finite element analysis of an integrally damped structure:

1) What type of element(s) should be used in modeling a sandwich structure?
2) What form should the discretized equations of motion take?
3) How should the equations of motion be solved?

Current finite element methods for analysis of damped structures can be placed in one of three categories based on how the second of these questions is answered. These methods are briefly reviewed in this section along with the advantages and disadvantages of each for design purposes.

Complex Eigenvalue Method

Suppose the discretized equations of motion take the form

$$M\ddot{x}+C\dot{x}+Kx=l(t) \tag{1}$$

where

M,C,K = physical coordinate mass, damping, and stiffness matrices (all real and constant)
$x,\dot{x},\ddot{x}$ = vectors of nodal displacements, velocities, and accelerations
l = vector of applied node loads

The solution can be carried out in terms of damped normal modes.[6,7] Both the eigenvalues and eigenvectors will in general be complex, but the method is nonetheless quite standard in that the modes obey an orthogonality condition and thus allow uncoupled equations of motion to be obtained.

There are two important drawbacks to the complex eigenvalue method. It is computationally expensive, typically three times the cost of the corresponding undamped eigensolution.[8] Also, for a structure to be described by Eq. (1), its materials, including any viscoelastic materials, must exhibit dynamic stress-strain behavior of a certain type. Storage moduli must be constant and loss moduli must increase linearly with frequency.[9] Real viscoelastic materials simply do not behave in such an accommodating way. Storage moduli tend to increase monotonically with frequency whereas loss factors exhibit a single, mild peak.[10]

Another finite element procedure that is often referred to as a complex eigenvalue method is to suppress the $C\dot{x}$ term in Eq. (1) and treat the stiffness matrix K as complex. This procedure is discussed later in terms of an example.

Direct Frequency Response Method

If the applied load varies sinusoidally in time, energy dissipation in the structure can be accounted for by treating the elastic constants of any or all the materials as complex quantities that are functions of frequency and temperature. These material properties are presumably available from sinusoidal tests. If the structure is linear, its response will be sinusoidal at the driving frequency, and the steady-state equations of motion will have the form:

$$[-M\omega^2 + K_1(\omega) + iK_2(\omega)]X(\omega) = L(\omega) \qquad (2)$$

where

$K_1(\omega), K_2(\omega)$ = stiffness matrices calculated using the real and imaginary parts of the material properties, respectively
ω = radian frequency of excitation
$L(\omega), X(\omega)$ = complex amplitude vectors of applied node loads and responses, respectively
$i = \sqrt{-1}$

It should be clear that material constants are truly complex quantities only in the sense that complex arithmetic is used simply as a convenient method of keeping track of relative phases under sinusoidal excitation.

There are several drawbacks to the direct frequency response method. It is computationally expensive because a general sinusoidal solution requires that the displacement impedance matrix [the bracketed quantity in Eq. (2)] be recalculated, decomposed, and stored at each of many frequencies. Further, the method does not give information of direct use to a designer in improving performance of a candidate structure.

The costliness of the direct frequency response method indicated by Eq. (2) is a direct result of the restriction to physical coordinates (as opposed to modal coordinates). This restriction is, in turn, caused by the form of the corresponding time domain representation. General convolution integral relations between forces and displacements must be admitted in order to accommodate the variation of K_1 and K_2 with frequency that is observed in real viscoelastic materials. Because not even the form, let alone the parameter values, of the convolution relation is generally known, it must be represented numerically in the frequency domain in terms of its Fourier transform. A tabular frequency representation can be arbitrarily accurate as long as the underlying stress-strain operator is linear. Use of such a data format is bound to be costly, however, particularly if a high level of frequency resolution is required.

Modal Strain Energy Method

In this approach it is assumed that the damped structure can be represented in terms of the real normal modes of the associated undamped system if appropriate damping terms are inserted into the uncoupled modal equations of motion. That is,

$$\ddot{\alpha}_r + \eta^{(r)}\omega_r\dot{\alpha}_r + \omega_r^2\alpha_r = l_r(t) \qquad (3)$$

$$x = \Sigma\phi^{(r)}\alpha_r(t) \qquad r = 1,2,3... \qquad (4)$$

where

α_r = rth modal coordinate
ω_r = natural radian frequency of the rth mode
$\phi^{(r)}$ = rth mode shape vector of the associated undamped system
$\eta^{(r)}$ = loss factor of the rth mode

It is implied that the physical coordinate damping matrix C of Eq. (1) need not be explicitly calculated but that it can be diagonalized, at least approximately, by the same real modal matrix that diagonalizes K and M.

The modal loss factors are calculated by using the undamped mode shapes and the material loss factor for each material. This general approach was first suggested by Kerwin and Ungar[11] in 1962. Its application by finite element methods was suggested by Rogers[12] and Medaglia.[13] For sandwich structures, the material loss factor of the metal face sheets is very small compared with that of the viscoelastic core. In this situation, the modal loss factor is found from

$$\eta^{(r)} = \eta_v [V_v^{(r)} / V^{(r)}] \qquad (5)$$

where η_v is the material loss factor of viscoelastic core evaluated at the rth calculated resonant frequency and $V_v^{(r)} / V^{(r)}$ is the fraction of elastic strain energy attributable to the sandwich core when the structure deforms in the rth mode shape.

A derivation of Eq. (5) is given in the Appendix. It is shown that modal loss factors obtained from Eq. (5) can be expected to approximate the computationally more expensive complex stiffness eigenvalue results. This result is also demonstrated by examples in a later section of this paper.

Calculation of the modal energy distributions fits quite naturally within finite element methods and is a standard option in some commercial codes.[14] The basic advantages of the method are that only undamped normal modes need be calculated and that the energy distributions are of direct use to the designer in deciding where to locate damping layers. The disadvantage is that some approximation is required to accommodate frequency-dependent material properties.

Finite Element Analysis of Three-Layer Sandwich Structures

Choice of Elements

Regardless of the solution method to be employed, modeling of sandwich structures requires that the strain energy due to shearing of the core be accurately represented. Practical considerations dictate that this be done with minimum increase in computation cost relative to a uniform, single-layer model. In this section, a modeling method is described that is reasonably efficient and has the important advantage of being readily implemented in MSC/NASTRAN, a widely available code.[8]

Figure 1 shows the arrangement for modeling of a three-layer sandwich. The face sheets are modeled with quadrilateral or triangular plate elements producing stiffness at two rotational and three translational degrees of freedom per node. The viscoelastic core is modeled with solid elements producing stiffness at three translational degrees of freedom per node. All nodes are at element corners. The plate elements are called TRIA3, QUAD4, TRIA6, and QUAD8, and the solid elements are called PENTA and HEXA in MSC/NASTRAN. A key feature of these plate elements in the present application is their ability to account for coupling between stretching and bending deformations.[15] This feature allows the plate nodes to be offset to one surface of the plate,

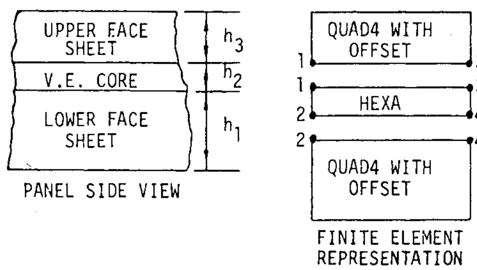


Fig. 1 Finite element modeling of a sandwich panel with viscoelastic core.

coincident with the corner nodes of the adjoining solid elements. In this way, a three-layer plate can be modeled with only two layers of nodes. Earlier methods implemented within NASTRAN were restricted to beams[16] or required four layers of nodes and extensive constraint equations to achieve the proper bending-shearing behavior of the sandwich.[17] Aspect ratios of the solid elements (in-plane dimension to thickness dimension) as high as 5000 have been used successfully to model the thin viscoelastic core layers. In the present analysis, Poisson's ratio of the core elements is taken to be 0.49.

Reduction of Equations of Motion

In all but the smallest problems, the mass and stiffness matrices are condensed by partitioning and Guyan reduction prior to calculation of eigenvalues. As usual in vibration analysis, some care is warranted in selection of the degrees of freedom to be retained during this reduction. For the sandwich structures studied in this paper, only out-of-plane displacements need be retained. Some displacements should be retained for both face sheets, although it is not necessary to keep both upper and lower face displacements at any single location on the model. This result is somewhat surprising in that virtually all sandwich panel theories assume the transverse displacements of the face sheets to be equal. If out-of-plane displacements of only one face sheet are kept, the results for natural frequency as well as core-to-total energy ratio can show a pronounced dependence on the Poisson's ratio of the core. Although such a dependence is probably real for some cases, such as doubly curved shells, it should not occur for simpler cases, such as straight sandwich beams —and in fact does not occur if the rule given above is observed in reducing the discretized equations of motion. Existing data on Poisson's ratio of most viscoelastic materials are probably not adequate for accurate modeling of doubly curved sandwich shells in the important transition region of the material and certainly not in the glassy region.

Solution Method

Once the model is assembled, either direct frequency response or modal strain energy analysis can be performed. In the direct frequency response analysis, the core material properties are input via a table as complex functions of frequency, and the solution proceeds as described earlier. In the modal strain energy method, a standard normal mode extraction run is made with all material constants treated as real and constant. The elastic strain energy in each element for each mode is calculated as well as the energy fraction in the viscoelastic core for each mode. These fractions multiplied by the core material loss factor give the modal loss factors, which are input via a damping-vs-frequency table for use in subsequent forced response calculations.

A basic difficulty with the modal strain energy method (or any normal mode method) is that the modal properties are obtained from system matrices that are assumed to be constant. Viscoelastic materials, however, have storage moduli that vary significantly with frequency. There is no theoretically correct way to resolve this contradiction. There are, however, great practical advantages to making response predictions in terms of a normal mode set obtained from constant material properties. This can be done with reasonable accuracy if a simple correction is made to the modal loss factors obtained by Eq. (5). This correction is only to the modal damping ratios because these are the only modal parameters that can be readily adjusted by the finite element analyst. The correction is obtained as follows.

For broadband excitation, most of the response of a given mode occurs within a narrow band around the mode's natural frequency. It is natural, then, to require that the energy distribution used to compute the loss factor for a given mode be obtained using a stiffness matrix evaluated for material properties taken at that mode's frequency. Because the natural frequencies themselves depend on material properties, an iterative solution of two simultaneous relations (the eigenvalue problem for each mode number and the material-property-vs-frequency relation) is required. This is readily done,[12] but a further problem remains. The final modal coordinate representation of the structure must come from a single stiffness matrix evaluated using a single value of storage modulus for the core material. Natural frequencies, mode shapes, and modal masses will be correct for, at most, one mode. A further correction of the modal loss factor has been found to give some improvement.

Each modal equation of motion has the form given in Eq. (3). At resonance the first and last terms on the left cancel each other. The response magnitude is inversely proportional to the product $\eta^{(r)}\omega_r$, which is the coefficient of the modal velocity. If $\eta^{(r)}$ is altered to correct for the error in ω_r, an improvement in peak response may be expected, although resonance will still occur at a slightly shifted frequency and some error will remain due to l_r, which depends on modal mass. In test cases run for sandwich beams,[12] it was found that taking ω_r to be proportional to $\sqrt{G_2}$ (G_2 = core shear modulus) would improve the agreement between the MSE method and the direct frequency response method. This is of course an approximation because ω_r depends on properties of the face sheets as well as the core. The modal damping ratios are adjusted according to

$$\eta^{(r)\prime} = \eta^{(r)}\sqrt{G_2(f_r)/G_{2,\text{ref}}} \tag{6}$$

where

$\eta^{(r)\prime}$ = adjusted modal damping ratio for the rth mode

$\eta^{(r)}$ = modal damping ratio for the rth mode obtained by iteration

$G_{2,\text{ref}}$ = core shear modulus used in final normal modes calculation to obtain modal frequencies, shapes, and masses

$G_2(f_r)$ = core shear modulus at $f=f_r$ where f_r is the rth mode frequency calculated with $G_2 = G_{2,\text{ref}}$

Examples

The modal strain energy method implemented in MSC/NASTRAN has been applied to a number of simple structural elements for which other solutions were available. Results are given in this section for sandwich beams, rings, and plates.

Sandwich Beams

Sandwich beams have been analyzed by a number of authors. DiTaranto[1] derived a sixth-order differential equation for vibration of a general three-layer beam. Rao[2] obtained complex eigenvalue solutions of this equation for a complex core shear modulus and various boundary conditions. It is convenient to use these results in that the complex eigenvalue solution yields modal loss factors that can be compared directly with those obtained by the MSE method. The assumptions of the sixth-order derivation are consistent with those of the finite element model except that, in the differential analysis, the rotations and out-of-plane displacements of the upper and lower face sheets are taken to be equal. This is quite reasonable for a thin core layer that tightly couples the face sheets.

A second, more widely known analytical solution is also available for three-layer beams.[18] It is based on the usual fourth-order differential equation for flexural vibration of a uniform beam but with the sandwich construction accounted for in terms of an equivalent complex bending stiffness. Core shear is not explicitly retained as a dependent variable and therefore cannot be prescribed at boundaries. Mode shapes are simply assumed to be sinusoidal, and general boundary conditions are not considered.

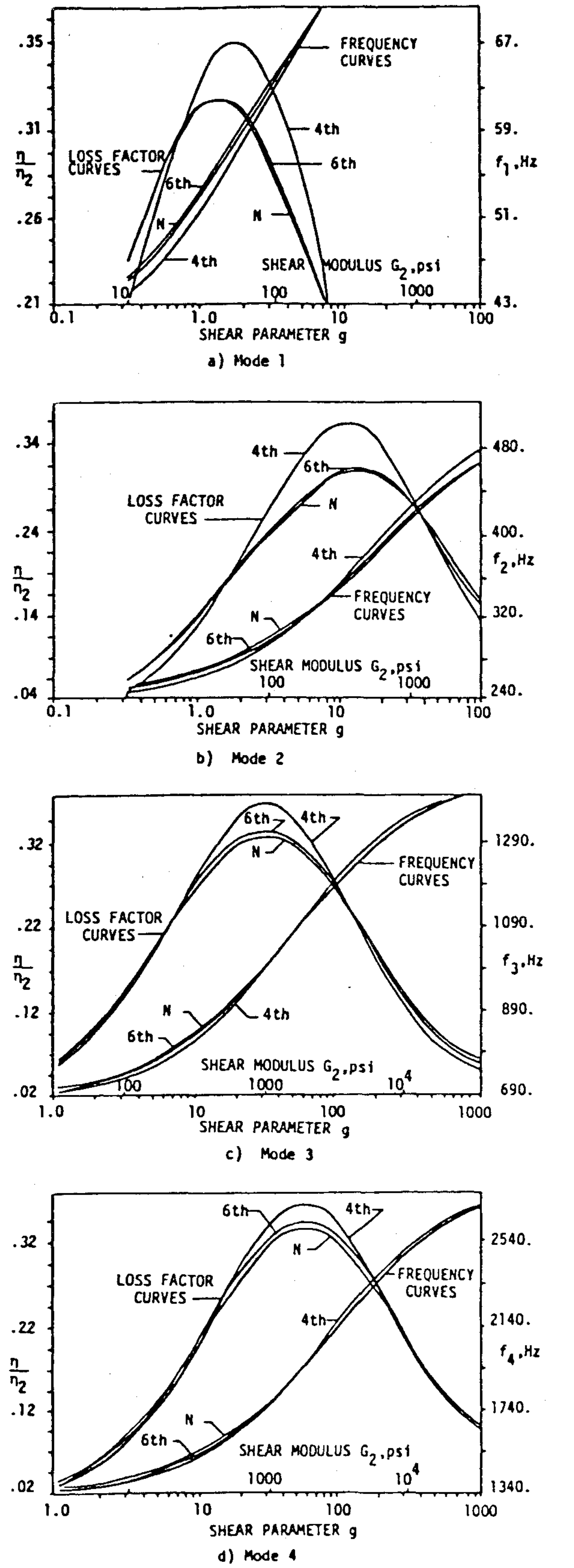


Fig. 2 Modal loss factors and natural frequencies of sandwich cantilever beam as computed by fourth-order theory, sixth-order theory, and NASTRAN.

Figure 2 shows a comparison between results for a cantilever sandwich beam as obtained from sixth-order theory, fourth-order theory,[18] and the MSE method using a NASTRAN model having 20 elements in the lengthwise direction. The 17.78 cm (7 in.) long cantilever beam has equal aluminum face sheets 1.52 mm (0.060 in.) thick and a viscoelastic core 0.127 mm (0.005 in.) thick. Results are given in terms of η/η_2, the composite loss factor normalized on the material loss factor of the core, and in terms of natural frequency for each of the first four modes. The ratio η/η_2 is obtained from the finite element MSE results simply as the ratio of core-to-total elastic strain energies [Eq. (5)] and thus does not require specification of η_2. The quantity η_2 is also called η_v in Eq. (5) and in the Appendix.

A value of core material loss factor much smaller than unity ($\eta_v = 0.01$) was used in obtaining the fourth-order and sixth-order results of Fig. 2. There is some doubt as to what agreement can be expected for higher values of η_v, although it appears to be adequate for engineering purposes even for loss factors in excess of unity. Further comparisons between finite element MSE and sixth-order differential results are given in Ref. 12 for core loss factors as high as 1.5. The results show a mild divergence for $\eta_v > 1$. A comparison of results from the MSE method and the complex eigenvalue method (i.e., constant complex stiffness), both implemented in NASTRAN, is shown in Table 1 for the beam geometry noted above and a core shear stiffness of 0.623 MPa (65 psi). It indicates that $\eta^{(r)}$ and η_v are almost exactly proportional over a wide range of η_v and agree quite well with MSE results. Computation cost for eigenvalue extraction in the complex stiffness, complex eigenvalue (CE) method in NASTRAN was found to be about five times greater than for real eigenvalue extraction in the corresponding MSE run. The Hessenberg method was used in the former and the Givens method in the latter.

Table 1 Comparison of normalized modal loss factors ($\eta^{(r)}/\eta_v$) for a sandwich beam calculated by the modal strain energy and complex stiffness eigenvalue methods

			CE	
Mode	Type[a]	MSE	$\eta_v = 0.3$	$\eta_v = 1.3$
1	1B	0.214	0.216	0.210
2	2B	0.280	0.287	0.276
3	1T	0.025	0.025	0.025
4	3B	0.213	0.223	0.218
5	4B	0.134	0.142	0.141
6	2T	0.026	0.026	0.026
7	5B	0.081	0.093	0.092
8	3T	0.031	0.031	0.031
9	6B	0.057	0.062	0.062

[a] nB = nth bending mode; nT = nth twisting mode.

The shear parameter g, shown as the abscissa in Fig. 2, is a real quantity that can be thought of as a normalized shear modulus for the core material:

$$g = \frac{G_2^*}{(1 + i\eta_2)} \frac{A_2 L^2}{t_2^2} \frac{(E_1 A_1 + E_3 A_3)}{E_1 A_1 E_3 A_3} \tag{7}$$

where G_2^* is the complex shear modulus of core material, η_2 the loss modulus of core material (η_v), A_2 the cross-sectional area of core, t_2 the thickness of core, L the beam length, E_1, E_3 the elastic moduli of face sheets, and A_1, A_3 the cross-sectional area of face sheets. The quantity $g(1 + i\eta_2)$ occurs as a coefficient in the nondimensional sixth-order differential equation of motion.[2]

Similar calculations were made for a variety of beam section geometries and boundary conditions, with similar results. The sixth-order and NASTRAN results for damping and frequencies were, for practical purposes, identical for the first six modes. Results from the fourth-order theory differed somewhat for certain boundary conditions, as would be expected based on the built-in assumptions.

Sandwich Rings

The modal strain energy method has been applied to several sandwich ring configurations for which Lu et al.[4] gave a closed-form solution as well as experimental results. The experimental data are in the form of driving point impedances for a single point load applied in the radial direction. These driving point impedances have been converted to admittance functions for comparison with results obtained by finite element modeling. Because the NASTRAN QUAD4 and HEXA elements are both capable of modeling curved surfaces, there is no basic difference in method between beams and rings. Some loss of accuracy could probably be expected at small radius to thickness ratios, although this situation was not investigated.

Figure 3 gives the dimensions and material properties for specimen #1 of Ref. 4. Figure 4 shows the experimental results for a sandwich ring having these dimensions and material properties. It can be noted that the viscoelastic material properties vary significantly over the 50-5000 Hz analysis band.

Finite element results obtained by three methods are shown in Fig. 4: 1) the direct frequency response method [Eq. (2)], which accounts exactly for material property variation with frequency; 2) the modal strain energy method with constant G_2, which allows for no variation; 3) the modal strain energy method with adjusted damping ratios, which allows an approximate accounting for property variations.

From Fig. 4 it can be seen that, with respect to damping, the unadjusted MSE method gives the best agreement with experiment. The adjusted MSE and direct frequency response methods agree very well with each other and show fair agreement with experiment. All three methods agree well with experiment in predicting resonant frequencies.

Unfortunately, the experimental data were not available in original form, and some inaccuracy was probably introduced in the several stages of replotting that were required. In addition, the material property relations used in the analyses are only fitted-curve approximations to data measured over 40-4000 Hz. The excellent agreement between experiment and the unadjusted MSE method, although encouraging, is thought to be at least partly coincidental. The unadjusted MSE method used constant values of $G_2 = 83.9$ MPa (12,164 psi) and $\eta_2 = 0.542$, corresponding to the variable properties evaluated at 500 Hz. These properties differ considerably from the actual values at other frequencies within the band.

The agreement between the direct frequency response and adjusted MSE methods is significant and is not subject to input parameter uncertainty. It is important because the MSE method is substantially less expensive and is more readily used in the design process. The simple correction applied to damping ratios is adequate, at least in this case, to account for the frequency dependence of core material properties.

It is recognized that comparison to experiment must be the final factor in judging the accuracy of any analysis method; however, as in the development of most finite element methods, it is more useful at first to employ closed-form solutions for comparison in order to bypass experimental uncertainties and expense. It is planned to use the analysis results of Lu et al.[4] to check finite element-MSE results for a variety of ring configurations and thereby uncover any potential problems in the finite element modeling of damped sandwich rings.

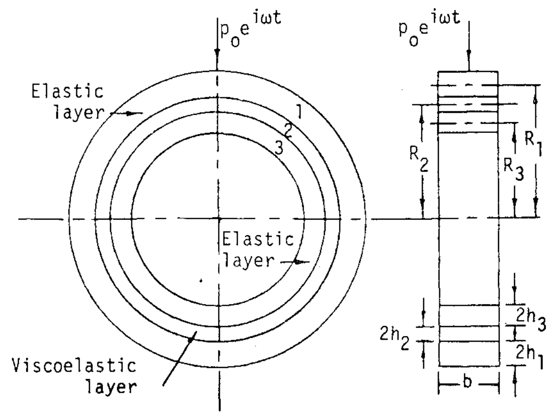
The comparison between finite element-MSE results and the analytical solution can also be made directly in terms of frequencies and modal loss factors. The loss factors are obtained from the admittance function solution of Lu et al.[4] by the usual half-power bandwidth method; that is,

$$\eta^{(r)} = (f_r' - f_r'')/f_r \tag{8}$$

where $\eta^{(r)}$ is the loss factor of rth mode, f_r is the resonant frequency of rth mode, and f_r', f_r'' are frequencies slightly above and below f_r at which the magnitude of the admittance function is reduced by 3 dB. Loss factors are obtained from the undamped finite element results by the modal strain energy method [Eq. (5)]. A comparison is given in Table 2 for the ring geometry shown in Fig. 3. Both solutions are for a hypothetical case of constant core material properties [$G_2 = 83.9$ MPa (12,164.0 psi), $\eta_2 = 0.542$].

Table 2 Natural frequencies and modal loss factors for a damped sandwich ring

	Analytical solution (Lu et al.[4])		NASTRAN/MSE	
Mode	Frequency (Hz)	Loss factor	Frequency (Hz)	Loss factor
1	660.7	0.0946	649.2	0.0974
2	1752.0	0.0446	1746.8	0.0501
3	3289.8	0.0271	3307.1	0.0320



R_1 = 13.02 cm $E_1 = E_3 = 2.88 \times 10^5$ MPa
$h_1 = h_3$ = .635 cm $\nu_1 = \nu_3 = 0.3$
h_2 = .549 cm b = 2.54 cm
$\rho_1 = \rho_3$ = 7.75 gm/cm^3 ρ_2 = 2.13 gm/cm^3
G_2 = exp[.50 ln(f) + 6.299] psi
η_2 = exp[.288 ln(f) - 2.402]

Fig. 3 Sandwich ring used for test case.

Sandwich Plates

The closed-form solution of Abdulhadi[3] for natural frequencies and modal loss factors of sandwich plates has been employed as a third test of the finite element-MSE method. His formulation is not valid for all boundary conditions but does hold when the plate is simply supported with unrestrained core shear at the edges. A 30.48×34.80 cm (12.0×13.69 in.) rectangular sandwich plate with these boundary conditions was analyzed using a NASTRAN model with 10 and 12 elements in the length and width directions, respectively. The upper and lower face sheet thicknesses were 0.762 mm (0.030 in.) and had the properties $E_l = 6.89 \times 10^4$ MPa (10^7 psi), $\nu = 0.30$, and $\rho_l = 2.74$ g/cm^3 (0.0988 lb/in.^{3}). The 0.254 mm (0.010 in.) thick core had the constant properties of $G_2 = 0.896$ MPa (130.0 psi), $\nu = 0.49$, $\rho_2 = 0.999$ g/cm^3 (0.0361 lb/in.^{3}), and $\eta_2 = 0.50$.

Hypothetical constant values were used for core shear modulus and loss factor in order that comparisons could be made for several modes with only a single NASTRAN run. The comparisons of the frequencies and modal loss factors for this plate, predicted by the NASTRAN/MSE and by the analytical method, are given in Table 3.

The agreement between NASTRAN and the differential equation solution for structural loss factor and frequency of the plate is excellent. These results are not surprising given the fact that, for these boundary conditions, the sixth-order sandwich plate theory is analogous to the sixth-order sandwich beam theory, which was known to agree well with finite element-MSE results.[12]

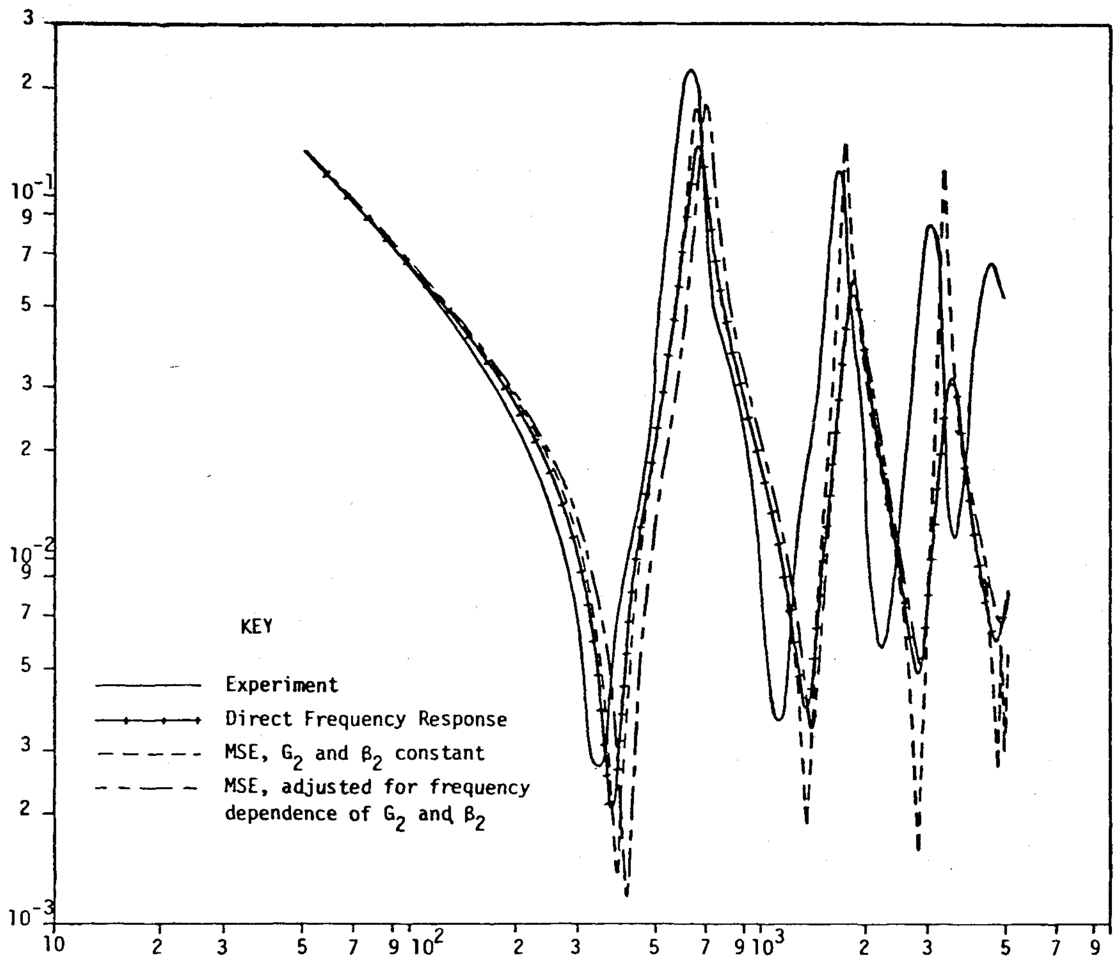


Fig. 4 Driving point velocity admittance of sandwich ring.

Table 3 Natural frequencies and modal loss factors for a rectangular, damped sandwich plate

	Analytical solution (Abdulhadi[3])		NASTRAN/MSE	
Mode	Frequency (Hz)	Loss factor	Frequency (Hz)	Loss factor
1	60.3	0.190	57.4	0.176
2	115.4	0.203	113.2	0.188
3	130.6	0.199	129.3	0.188
4	178.7	0.181	179.3	0.153
5	195.7	0.174	196.0	0.153

Conclusions

Methods for finite element analysis of structures with layered viscoelastic damping have been reviewed. The important features of each relative to design analysis have been noted. None of these methods is yet in widespread use for day-to-day design work. The modal strain energy method appears to be the most promising for large-scale applications. The method has been demonstrated on simple problems for three-layer beams, rings, and plates. Good agreement has been obtained with closed-form solutions for natural frequencies and modal loss factors. A simple empirical correction has been given that allows an approximate correction for frequency-dependent material properties.

Additional work is still required to make the MSE method a practical design tool. In particular, application to singly and doubly curved shells should be investigated. The effect of Poisson's ratio on a viscoelastic core should be studied. Further comparison to experiment will also be required.

Appendix: Relation Between Complex Eigenvalue Method and Modal Strain Energy Method

An approximate expression is derived in this Appendix for the modal loss factor obtained from an eigenvalue analysis of a structure with complex stiffness. The expression is identical to that obtained heuristically by Ungar[11] which forms the basis of the modal strain energy method.

The discretized (i.e., finite element) version of a partial differential equation for free vibration of a sandwich structure (or any structure) is:

$$M\ddot{x} + Kx = 0 \tag{A1}$$

where the stiffness matrix K is constant but complex if the structure contains a viscoelastic material. Equation (A1) is converted to an eigenvalue problem by assuming a solution of the form

$$X = \phi^{*(r)} e^{ip_r^* t} \tag{A2}$$

where p_r^* and $\phi^{*(r)}$ are the rth complex eigenvalue and eigenvector; that is, following Rao's notation,[2]

$$\phi^{*(r)} = \phi_R^{(r)} + i\phi_I^{(r)} \tag{A3}$$

$$p_r^* = p_r\sqrt{1 + i\eta^{(r)}} \tag{A4}$$

where $\phi_R^{(r)}$, $\phi_I^{(r)}$, $\eta^{(r)}$, and p_r are real. The term $\eta^{(r)}$ is the loss factor for the rth mode. The eigenvalue problem is then, from Eqs. (A1) and (A2):

$$K\phi^* = p^{*2}M\phi^* \tag{A5}$$

Now if K were purely real, $\phi^{*(r)}$ and p_r^* would be real and related by the usual Raleigh's quotient formula:

$$p_r^2 = [\phi^{(r)T} K \phi^{(r)}] / [\phi^{(r)T} M \phi^{(r)}] \quad \text{(A6)}$$

where the * superscript is dropped to denote a real quantity. If K is perturbed by δK, where δK is complex, p_r^2 will likewise acquire an imaginary part, which may be written as $i\eta p^2$ after Eq. (A4). Then, if the perturbed stiffness matrix is written as

$$K = K_R + iK_I \quad \text{(A7)}$$

the following is obtained from Eqs. (A4), (A6), and (A7), after dropping the mode index r:

$$p^2(1+i\eta) = \frac{\phi^{*T} K_R \phi^*}{\phi^{*T} M \phi^*} + i\frac{\phi^{*T} K_I \phi^*}{\phi^{*T} M \phi^*} \quad \text{(A8)}$$

An approximate value for η can be calculated by approximating the complex eigenvector ϕ^* by the real vector ϕ, which is calculated from purely elastic analysis, i.e., by suppressing the imaginary part of K. The approach is essentially an extension of Raleigh's principle into the complex domain. Making this approximation in Eq. (A8) and equating real and imaginary parts gives

$$p^2 = \phi^T K_R \phi / \Phi^T M \Phi \quad \text{(A9)}$$

$$p^2 \eta = \phi^T K_I \phi / \Phi^T M \Phi \quad \text{(A10)}$$

If the matrix K is obtained by finite element analysis, it may be divided into two additive terms. The first, called K_e, is obtained from contributions of the purely elastic elements (the plate elements used to model the face sheets). The second, called K_v, is obtained from the viscoelastic elements (the solid elements used to model the core). Both terms are matrices of the same order as K,

$$K = K_e + K_v \quad \text{(A11)}$$

K_e will be completely real. K_v will be complex but, for the present case where only a single viscoelastic material is involved, its imaginary and real parts will have the ratio η_v:1 where η_v is the material loss factor of the core. Then,

$$K_v = K_{vR} + iK_{vI} \quad \text{(A12)}$$

$$K_v = K_{vR}(1+i\eta_v) \quad \text{(A13)}$$

By previous assumption, only K_v contributes to K_I, so

$$K_I = K_{vI} \quad \text{(A14)}$$

When a purely elastic normal modes analysis is performed, the strain energy associated with a given mode shape is

$$V = \phi^T K_R \phi \quad \text{(A15)}$$

The portion of this energy that is attributable to strain in the core is

$$V_v = \phi^T K_{vR} \phi \quad \text{(A16)}$$

Eliminating p^2 between Eqs. (A8) and (A9) gives

$$\eta = \eta_v (\phi^T K_I \phi / \phi^T K_R \phi) \quad \text{(A17)}$$

Combining Eqs. (A13-A17) and reinstating the mode index superscript gives the final result for modal loss factor in terms of elastic energies,

$$\eta^{(r)} = \eta_v (V_v^{(r)} / V^{(r)}) \quad \text{(A18)}$$

This derivation is intended to motivate and clarify the comparison of results from complex eigenvalue analysis and modal strain energy analysis. It should be noted, however, that the problem statement itself is not entirely realistic. It is well known[19] that complex stiffnesses that do not vary with frequency lead to system responses that are noncausal (response anticipates input) and hence are not physically realizable. Nonetheless, the comparison is believed to be useful in that the symptoms of noncausality are quite weak[19] and the identical assumptions are applied in both methods.

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